

# ALMOST KÄHLER 4-MANIFOLDS WITH $J$ -INVARIANT RICCI TENSOR AND SPECIAL WEYL TENSOR

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## 1. INTRODUCTION

An *almost Kähler structure* on a manifold  $M^{2n}$  is a triple  $(g, J, \Omega)$  of a Riemannian metric  $g$ , an orthogonal almost complex structure  $J$ , and a symplectic form  $\Omega$  satisfying the compatibility relation

$$\Omega(X, Y) = g(JX, Y), \quad (1)$$

for any tangent vectors  $X, Y$  to  $M$ . If the almost complex structure  $J$  is integrable we obtain a *Kähler structure*. Many efforts have been done in the direction of finding curvature conditions on the metric which insure the integrability of the almost complex structure. A famous conjecture of Goldberg [26] states that a compact almost Kähler, Einstein manifold is in fact Kähler. Important progress was made by K. Sekigawa who proved that the conjecture is true if the scalar curvature is non-negative [46]. The case of negative scalar curvature is still wide open, although recently there has been some significant progress in dimension 4, see [7, 8, 9, 42, 43, 44]. It is now known that if the conjecture turns out to be true, the compactness should play an essential role. Nurowski and Przanowski [43] constructed a 4-dimensional, local example of Einstein, strictly almost Kähler manifold; this was generalized by K. Tod (see [8, 9]) to give a family of such examples. It is interesting to remark that the structure of the Weyl tensor of all these examples is unexpectedly special. An important recent result of J. Armstrong [8, 9] states that any 4-dimensional almost Kähler, non-Kähler, Einstein manifold is obtained by Tod's construction, provided that the Kähler form is an eigenform of the Weyl tensor. In the compact case, on the other hand, some of the positive partial results on the conjecture in dimension 4 have been obtained exactly by imposing some additional assumptions on the structure of the Weyl tensor ([7, 8, 9, 44]).

The recently discovered Seiberg-Witten invariants could have an impact towards a complete answer to the Goldberg conjecture in dimension 4. These are invariants of smooth, oriented, compact 4-dimensional manifolds. The works of Taubes [47, 48, 49] and others showed the invariants to be particularly interesting for symplectic 4-manifolds. (See also [34] for a quick introduction to Seiberg-Witten invariants and some of their applications to symplectic geometry.) LeBrun [37], [38], [39] and Kotschick [35] have also

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proved several interesting applications of the Seiberg-Witten theory to the existence of Einstein metrics in dimension 4, which shed some light on the Goldberg conjecture as well.

In this paper we will not directly work on the Goldberg conjecture, but rather on problems which are parallel to it. The curvature of a Kähler metric has strong symmetry properties with respect to  $J$ . On the other hand, an arbitrary almost Hermitian (or almost Kähler) metric may not have any of these symmetry properties. A. Gray [25] introduced classes of almost Hermitian manifolds whose curvature tensor has a certain degree of resemblance to that of a Kähler manifold (see Section 3). The condition that the Ricci tensor is  $J$ -invariant could be considered to be the minimal degree of resemblance. This is weaker than the Einstein condition and may be even more natural in the context of almost Hermitian geometry, as some interesting variational problems on compact symplectic manifolds have lead to almost Kähler metrics with  $J$ -invariant Ricci tensor. It was shown in [14] that they are the critical points of the Hilbert functional, the integral of the scalar curvature, restricted to the set of all compatible metrics to a given symplectic form. Compatible Kähler metrics provide absolute maxima for the functional in this setting [13]. Blair and Ianus asked if the  $J$ -invariance of the Ricci tensor is a sufficient condition for the integrability of an almost Kähler structure on a compact manifold [14]. If true, this would be a stronger statement than the Goldberg conjecture. However, 6-dimensional counter-examples to the question of Blair and Ianus were found by Davidov and Muškarov [17] (see also [1]). Multiplying these by compact Kähler manifolds, one obtains counter-examples in any dimension  $2n, n \geq 3$ . No example of a strictly almost Kähler structure with  $J$ -invariant Ricci tensor is known yet on a compact 4-dimensional manifold.

The examples of [17] have, in fact, a higher degree of resemblance to Kähler structures than just the  $J$ -invariant Ricci tensor. Indeed, they are strictly almost Kähler manifolds satisfying the second curvature condition of Gray (see Section 3 for definition and [25]). In dimension 4, the second curvature condition of Gray on an almost Hermitian metric just means that the Ricci tensor and the positive Weyl tensor have the same symmetries as for a Kähler metric. The integrability of the almost complex structure follows even locally for an Einstein, almost Kähler 4-dimensional manifold which satisfies the second curvature condition of Gray [7, 9, 44]. The main result of this paper is that compact, 4-dimensional, *strictly* almost Kähler manifolds satisfying the second curvature condition of Gray could exist only under strong topological assumptions:

**Theorem 1.** *A compact, 4-dimensional, almost Kähler manifold which satisfies the second curvature condition of Gray is Kähler, provided that either the Euler characteristic or the signature of the underlying manifold does not vanish.*

Moreover, if there exists a compact, 4-dimensional, non-Kähler, almost Kähler manifold  $(M, g, J)$  satisfying the second curvature condition of Gray, then  $(M, g)$  admits an orthogonal Kähler structure  $\bar{J}$ , compatible with the reversed orientation of  $M$ , such that  $(M, \bar{J})$  is a minimal class VI complex surface.

**Note:** Some time after this paper was completed, the authors in collaboration with D. Kotschick proved that the exceptional case of the Theorem 1 cannot occur (*An integrability theorem for almost Kähler 4-manifolds*, C. R. Acad. Sci. Paris, t. 329, sér. I (1999), 413-418). Thus, any compact almost Kähler 4-manifold which satisfies the second curvature condition of Gray is, in fact, Kähler.

Note the contrast with the Hermitian case. For Hermitian surfaces the second curvature condition of Gray is equivalent to the  $J$ -invariance of the Ricci tensor; a number of compact, non-Kähler Hermitian surfaces with  $J$ -invariant Ricci tensor have been provided on rational surfaces ([4]).

Another natural problem which involves the  $J$ -invariance of the Ricci tensor is the description of the almost Kähler manifolds  $(M, g, J, \Omega)$  whose sectional curvature at any point is the same on all totally-real planes. We will refer to these manifolds as almost Kähler manifolds of *pointwise constant totally-real sectional curvature*. Recall that a two-plane  $\sigma = X \wedge Y$  in the tangent bundle  $TM$  is said to be *totally-real* if  $J\sigma = JX \wedge JY$  is orthogonal to  $\sigma$ . It was shown in [21, 22] that almost Kähler manifolds of pointwise constant totally-real sectional curvature of dimension higher than 4 are in fact complex space forms. To our knowledge, the 4-dimensional case which we consider here is still open. We also note that it has been open for some time the question of existence of a strictly almost Kähler structure of constant sectional curvature in dimension 4. Now the answer is known to be negative, as a consequence of results of [8, 44].

The condition that  $\sigma = X \wedge Y$  is a totally-real plane is, in fact, equivalent to  $g(JX, Y) = 0$ , i.e., to  $\sigma$  being a Lagrangian plane with respect to the fundamental 2-form  $\Omega$ . Thus, the space of totally-real two-planes is determined by  $\Omega$  and it does not depend on the choice of an  $\Omega$ -compatible almost Kähler structure. Given a symplectic 4-manifold  $(M, \Omega)$ , it is natural to ask if there are any  $\Omega$ -compatible almost Kähler structures which have pointwise constant totally-real sectional curvature. It is easily checked that for any such a structure the Ricci tensor is  $J$ -invariant and the Weyl tensor is self-dual (Lemma 5 below). In the compact case we conjecture that *any compact, 4-dimensional almost Kähler manifold of pointwise constant totally-real sectional curvature is, in fact, a self-dual Kähler surface*. The compact self-dual Kähler surfaces are completely described by [16] (see also [15], [18], [28]) — they are all Riemannian locally symmetric spaces and in particular have constant scalar curvature. We are able to prove the following:

**Theorem 2.** *Any compact almost Kähler 4-manifold of pointwise constant totally-real sectional curvature and of constant scalar curvature is Kähler.*

Note also that the complete classification of compact Hermitian surfaces of (pointwise) constant totally-real sectional curvature was recently obtained in [6, 3].

The paper is organized as follows:

Section 2 is devoted to local considerations on almost Hermitian manifolds with  $J$ -invariant Ricci tensor. Many of our results here could be considered as further development of the so called “*Riemannian Goldberg-Sachs theory*” [4], with applications to almost Kähler 4-manifolds (see subsection 2.2). In particular, we obtain in Corollary 1 that *any almost Kähler anti-self-dual 4-manifold with  $J$ -invariant Ricci tensor is Kähler scalar-flat*. Slightly stronger results are obtained computing the Bach tensor of an almost Kähler 4-manifold with  $J$ -invariant Ricci tensor (see Lemma 3 in subsection 2.3 and its corollary). The technical part of our work uses the  $U(2)$ -decomposition of the curvature, first and second Bianchi identities and some Weitzenböck formulas.

In Section 3 we specify the local considerations of the previous section to the case of almost Kähler 4-manifolds satisfying the second Gray condition. We obtain that if such a manifold  $(M, g, J, \Omega)$  is *not* Kähler, then it admits a *negative* almost Kähler structure  $(\bar{M}, g, \bar{J}, \bar{\Omega})$ , which has quite strong symmetry properties of the curvature as well (Proposition 2). Imposing also compactness, the proof of our main result, Theorem 1, is done in several steps: Assuming that the given almost Kähler structure is not Kähler, we first use some results from the Seiberg-Witten theory to prove that the Euler number and the signature of the manifold both vanish. Then we show that the negative almost Kähler structure must be Kähler, and using the Kodaira classification we conclude that the corresponding complex surface belongs to class VI.

Section 4 is devoted to almost Kähler 4-manifolds of (pointwise) constant totally-real sectional curvature. The proof of Theorem 2 is an application of the results of Section 2 and [19, 9].

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**Dedication:** We would like to dedicate this paper to the memory of Alfred Gray, whom we did not know personally, but whose mathematical work inspired us.

## 2. ALMOST HERMITIAN 4-MANIFOLDS WITH $J$ -INVARIANT RICCI TENSOR

**2.1. The  $U(2)$ -decomposition of the curvature tensor of an almost Hermitian 4-manifold.** Let  $(M, g)$  be a 4-dimensional, oriented Riemannian manifold. The Hodge operator  $*$ , acting as an involution on the bundle of 2-forms, induces the orthogonal splitting:

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

where  $\Lambda^+ M$ ,  $(\Lambda^- M)$ , is the bundle of self-dual, (anti-self-dual) 2-forms. We will freely identify vectors and co-vectors via the metric  $g$  and, accordingly, a tensor  $\phi \in T^* M^{\otimes 2}$  with the corresponding endomorphism of the tangent bundle  $TM$ , by  $\phi(X, Y) = g(\phi(X), Y)$ . Also, if  $\phi, \psi \in T^* M^{\otimes 2}$ , by  $\phi \circ \psi$  we understand the endomorphism of  $TM$  obtained by the composition of the endomorphisms corresponding to the two tensors.

Considering the Riemannian curvature tensor  $R$  as a symmetric endomorphism of  $\Lambda^2 M$ , we have the following well-known  $SO(4)$ -decomposition:

$$R = \frac{s}{12} id + \tilde{Ric}_0 + W^+ + W^-, \quad (2)$$

where:

- $s$  is the scalar curvature;
- $\tilde{Ric}_0$  is the Kulkarni-Nomizu extension of the traceless Ricci tensor,  $Ric_0$ , to an endomorphism of  $\Lambda^2 M$  anti-commuting with  $*$ ;
- $W^\pm$  are respectively the self-dual and anti-self-dual parts of the Weyl tensor  $W$ .

We view  $W^\pm$ , respectively, as sections of the bundles  $\mathcal{W}^\pm = Sym_0(\Lambda^\pm M)$  of symmetric, traceless endomorphisms of  $\Lambda^\pm M$ . The manifold  $(M, g)$  is said to be self-dual (anti-self-dual), if  $W^- = 0$  ( $W^+ = 0$ ). The manifold is Einstein if  $\tilde{Ric}_0$  (or, equivalently,  $Ric_0$ ) vanishes identically.

Let  $(M, g, J, \Omega)$  be an almost Hermitian, 4-dimensional manifold, i.e., a 4-dimensional smooth manifold endowed with a triple  $(g, J, \Omega)$  of a Riemannian metric  $g$ , an orthogonal almost complex structure  $J$  and a non-degenerate 2-form, not necessarily closed, satisfying the compatibility relation (1). The action of the almost complex structure  $J$  extends to the cotangent bundle  $T^* M$ , by

$$(J\alpha)(X) = -\alpha(JX),$$

and to the bundle of 2-forms  $\Lambda^2 M$ , by

$$(J\phi)(X, Y) = \phi(JX, JY).$$

Because of this action, we have the following splitting of the bundle of (real) 2-forms.

$$\Lambda^2 M = \Lambda^{inv} M + \Lambda^{anti} M = \mathbf{R}\Omega + \Lambda_0^{inv} M + \Lambda^{anti} M,$$

where  $\Lambda^{inv} M$ ,  $\Lambda^{anti} M$  are respectively the spaces of  $J$ -invariant and  $J$ -anti-invariant 2-forms and  $\Lambda_0^{inv} M$  is the space of trace-free  $J$ -invariant 2-forms. Note that  $\Lambda^{anti} M$  is the real underlying bundle of the anti-canonical bundle  $(K_J)^{-1} := \Lambda^{0,2} M$  of  $(M, J)$ ; the induced complex structure  $\mathcal{J}$  on  $\Lambda^{anti} M$  is

then given by  $\mathcal{J}\phi(.,.) := -\phi(J.,.)$ . The vector bundle  $\Lambda^{inv}M$  of  $J$ -invariant real 2-forms is the real underlying bundle of the bundle of (1,1) forms.

Whereas the above decomposition holds in any dimension, in dimension 4 it links very nicely with the self-dual, anti-self-dual decomposition of 2-forms given by the Hodge star operator:

$$\Lambda^+M = \mathbf{R}\Omega \oplus \Lambda^{anti}M, \quad (3)$$

$$\Lambda^-M = \Lambda_0^{inv}M. \quad (4)$$

Because of the splitting (3), the bundle  $\mathcal{W}^+$  decomposes into the sum of three sub-bundles,  $\mathcal{W}_1^+$ ,  $\mathcal{W}_2^+$ ,  $\mathcal{W}_3^+$  defined as follows, see [50]:

- $\mathcal{W}_1^+ = \mathbf{R} \times M$  is the sub-bundle of elements preserving the decomposition (3) and acting by homothety on the two factors; hence is the trivial line bundle generated by the element  $\frac{1}{8}\Omega \otimes \Omega - \frac{1}{12}id^+$ .
- $\mathcal{W}_2^+ = \Lambda^{anti}M$  is the sub-bundle of elements which exchange the two factors in (3); the real isomorphism with  $\Lambda^{anti}M$  is seen by identifying each element  $\phi$  of  $\Lambda^{anti}M$  with the element  $\frac{1}{2}(\Omega \otimes \phi + \phi \otimes \Omega)$  of  $\mathcal{W}_2^+$ .
- $\mathcal{W}_3^+ = Sym_0(\Lambda^{anti}M)$  is the sub-bundle of elements preserving the splitting (3) and acting trivially on the first factor  $\mathbf{R}\Omega$ .

Consequently, we have the following splitting of the Riemannian curvature operator  $R$  ([50]):

$$R = \frac{s}{12}id + (\tilde{Ric}_0)^{inv} + (\tilde{Ric}_0)^{anti} + W_1^+ + W_2^+ + W_3^+ + W^-, \quad (5)$$

where  $(\tilde{Ric}_0)^{inv}$  and  $(\tilde{Ric}_0)^{anti}$  are the Kulkarni-Nomizu extensions of the  $J$ -invariant and the  $J$ -anti-invariants parts of the traceless Ricci tensor, respectively, and  $W_i^+$  are the projections of  $W^+$  on the spaces  $\mathcal{W}_i^+$ ,  $i = 1, 2, 3$ . The component  $W_1^+$  is given by

$$W_1^+ = \frac{\kappa}{8}\Omega \otimes \Omega - \frac{\kappa}{12}id^+, \quad (6)$$

where  $\kappa$  is the so called *conformal scalar curvature* of  $(g, J)$ , defined by

$$\kappa = 3 \langle W^+(\Omega), \Omega \rangle,$$

where  $\langle ., . \rangle$  denotes the extension of the Riemannian metric  $g$  to the bundle  $\Lambda^2M$ , so that the norm induced by  $\langle ., . \rangle$  to be half of the usual tensor norm on  $\Lambda^1M \otimes \Lambda^1M$ . The component  $W_2^+$  is determined by the skew-symmetric part of the *star-Ricci tensor*, defined for an almost Hermitian manifold by:

$$Ric^*(X, Y) = -R(\Omega)(JX, Y).$$

We thus get

$$W_2^+ = -\frac{1}{4}(\Psi \otimes \Omega + \Omega \otimes \Psi), \quad (7)$$

where  $-\frac{1}{2}J\Psi$  is the skew symmetric part of  $Ric^*$ . Note that in general, the star-Ricci tensor is neither symmetric nor skew-symmetric. The *star-scalar*

curvature  $s^*$  is the trace of  $Ric^*$ . It is easy to check the following relation between the scalar curvatures of a 4-dimensional almost Hermitian manifold:

$$\kappa = \frac{1}{2}(3s^* - s). \quad (8)$$

On any almost Kähler manifold (of arbitrary dimension), the difference between the star-scalar curvature and the scalar curvature “measures” the integrability of the almost complex structure:

$$s^* - s = \frac{1}{2}|\nabla J|^2. \quad (9)$$

Let us also remark here that  $(\tilde{Ric}_0)^{anti}$  determines the part of the curvature acting from  $\Lambda^{anti}M$  to  $\Lambda^-M$ , thus a 4-dimensional almost Hermitian manifold has  $J$ -invariant Ricci tensor if and only if

$$\langle R(\Lambda^{anti}M), \Lambda^-M \rangle = 0.$$

**2.2. The second Bianchi identity for almost Hermitian 4-manifolds with  $J$ -invariant Ricci tensor.** The co-differential  $\delta W^+$  of the positive Weyl tensor of  $(M, g)$  is a section of the rank 8 vector bundle

$$\mathcal{V} = Ker(tr : \Lambda^1 M \otimes \Lambda^+ M \longrightarrow \Lambda^1 M),$$

where  $tr$  is defined by  $tr(\alpha \otimes \phi) = \phi(\alpha)$  on decomposed elements. The vector bundle  $\mathcal{V}$  splits as  $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ , see [4], where:

(a)  $\mathcal{V}^+$  is identified with the cotangent bundle  $T^*M$  by

$$T^*M \ni \alpha \mapsto A = J\alpha \otimes \Omega - \frac{1}{2} \sum_{i=1}^4 e_i \otimes (\alpha \wedge e_i - J\alpha \wedge J e_i), \quad (10)$$

$$\mathcal{V}^+ \ni A \mapsto \alpha = -\frac{1}{2}J \langle A, \Omega \rangle,$$

where  $\langle A, \Omega \rangle$  denotes the 1-form defined by  $X \mapsto \langle A_X, \Omega \rangle$ .

(b)  $\mathcal{V}^-$  is identified, as a real vector bundle, to the real, rank 4, vector bundle  $\Lambda^{0,1}M \otimes K_J^{-1}$ ; if  $\phi$  is a non-vanishing local section of  $\Lambda^{anti}M$ , then it trivializes  $K_J^{-1}$  and  $\mathcal{V}^-$  can be again identified with  $T^*M$  by

$$\begin{aligned} \beta \in T^*M &\mapsto B = \sum_{i=1}^4 e_i \otimes (J\beta \wedge \phi(e_i) + \beta \wedge \mathcal{J}\phi(e_i)) \\ B \in \mathcal{V}^- &\mapsto \beta = -\frac{1}{|\phi|^2}J \langle B, \phi \rangle. \end{aligned} \quad (11)$$

We denote by  $(\delta W^+)^+$ , resp.  $(\delta W^+)^-$ , the component of  $\delta W^+$  on  $\mathcal{V}^+$ , resp. on  $\mathcal{V}^-$ . The *Cotton-York tensor*  $C$  of  $(M, g)$  is defined by:

$$C_{X,Y,Z} = \frac{1}{2} \left[ \nabla_Z \left( \frac{s}{12}g + Ric_0 \right) (Y, X) - \nabla_Y \left( \frac{s}{12}g + Ric_0 \right) (Z, X) \right].$$

The second Bianchi identity reads as

$$\delta W = C, \quad (12)$$

where  $\delta W$  is the co-differential of the Weyl tensor  $W$ . In particular, we have

$$\delta W^+ = C^+, \quad (13)$$

where  $C^+$  denotes the self-dual part of  $C_X$ ,  $X \in TM$ .

For an almost Hermitian 4-manifold  $(M, g, J, \Omega)$ , denote by  $\theta$  the Lee 1-form given by  $\theta = J\delta\Omega$ . Then we have the following:

**Lemma 1.** *Let  $(M, g, J, \Omega)$  be an almost Hermitian 4-manifold with  $J$ -invariant Ricci tensor. Denote by  $\alpha$  the 1-form corresponding to  $(\delta W^+)^+$  via the isomorphism (10). Then  $\alpha$  is given by*

$$\alpha = -\frac{ds}{12} + \frac{1}{2}Ric_0(\theta).$$

*Proof:* From (13) we have

$$\begin{aligned} \alpha(X) &= -\frac{1}{4} \sum_{i=1}^4 \nabla_{e_i} \left( \frac{s}{12} g + Ric_0 \right) (J e_i, J X) = \\ &= -\frac{1}{4} \left[ \frac{ds}{12}(X) - (\delta Ric_0)(X) + \sum_{i=1}^4 Ric_0(e_i, J(\nabla_{e_i} J)(X)) - Ric_0(\theta, X) \right] = \\ &= -\frac{1}{4} \left[ \frac{ds}{3}(X) + \sum_{i=1}^4 Ric_0(e_i, J(\nabla_{e_i} J)(X)) - Ric_0(\theta, X) \right]. \end{aligned}$$

For any almost Hermitian manifold  $\nabla J$  is given by (cf. e.g. [31]):

$$\nabla_X J = \frac{1}{2}(X \wedge J\theta + JX \wedge \theta) + \frac{1}{2}N_{JX},$$

where  $N_X(\cdot, \cdot) = g([J, J] - [\cdot, \cdot] - J[J, \cdot] - J[\cdot, J], X)$  is the Nijenhuis tensor of  $(M, J)$ . Using this formula and the fact that the Ricci tensor is  $J$ -invariant, we compute

$$\sum_{i=1}^4 Ric_0(e_i, J(\nabla_{e_i} J)(X)) = -Ric_0(\theta, X),$$

and we obtain the expression claimed for  $\alpha$ .  $\square$

Regarding the component  $(\delta W^+)^-$ , let  $\phi$  be a (locally defined) section of  $\Lambda^{anti}M$ , with  $|\phi|^2 = 2$ , and denote by  $\beta$  the 1-form corresponding to  $(\delta W^+)^-$  via the isomorphism (11). It follows from (10) and (11) that

$$\beta = \frac{1}{2} \left( -J \langle \delta W^+, \phi \rangle - \frac{1}{2} \mathcal{J}\phi(\alpha) \right).$$

To compute  $J \langle \delta W^+, \phi \rangle$  we will proceed in the same way as computing  $J \langle \delta W^+, \Omega \rangle$  in the proof of Lemma 1; we will now consider instead of  $J$  the almost complex structure  $I$  whose Kähler form is  $\phi$ . Observe that



$Ric_0$  is now  $I$ -anti-invariant. Involving Lemma 1, together with the following expressions for the covariant derivatives of  $\nabla J$  and  $\nabla I$ :

$$\nabla J = a \otimes I + b \otimes (J \circ I);$$

$$\nabla I = -a \otimes J + c \otimes (J \circ I),$$

where  $a, b, c$  are some 1-forms, we eventually obtain

$$\beta = -\frac{1}{4} Ric_0(a + Jb). \quad (14)$$

Note that the almost complex structure  $J$  is integrable (resp. almost Kähler) iff  $b = Ja$  (resp.  $b = -Ja$ ); we thus obtained the following:

**Lemma 2.** ([4, Proposition 4]) *Let  $(M, g, J, \Omega)$  be an almost Hermitian 4-manifold with  $J$ -invariant Ricci tensor. Suppose that the traceless Ricci tensor of  $g$  does not vanish. Then  $J$  is integrable if and only if*

$$(\delta W^+)^- = 0.$$

**Corollary 1.** *Any anti-self-dual almost Kähler 4-manifold of  $J$ -invariant Ricci tensor is Kähler (hence, scalar flat).*

*Proof:* Suppose that the traceless Ricci tensor does not vanish on an open subset  $U$  of  $M$ . It follows from Lemma 2 that  $(g, J, \Omega)$  is Kähler on  $U$ . Hence the scalar curvature vanishes on  $U$  and, by continuity, this holds on the closure of  $U$ . If  $M - \bar{U}$  is non-empty, then  $(g, J, \Omega)$  is an Einstein, anti-self dual, almost Kähler structure on the open set  $M - \bar{U}$ . By [8, Theorem 2.4], or by [44, Theorem 1], it follows that  $(g, J, \Omega)$  is Kähler on  $M - \bar{U}$ . Thus the scalar curvature is zero everywhere on  $M$  (see (6), (8) and (9)), hence  $(g, J, \Omega)$  is a Kähler structure on  $M$ .  $\square$

**2.3. The Bach tensor of almost Hermitian 4-manifolds with  $J$ -invariant Ricci tensor.** The Bach tensor  $B$  of a (4-dimensional) Riemannian manifold  $(M, g)$  is defined by

$$B_{X,Y} = \sum_{i=1}^4 [\nabla_{e_i}(\delta W)(X, e_i, Y) + W(X, e_i, h(e_i), Y)], \quad (15)$$

where  $W$  is the Weyl conformal tensor and  $h = \frac{1}{2}(Ric - \frac{s}{6}g)$  is the normalized Ricci tensor of  $(M, g)$ . It is known that on a compact 4-manifold  $B_{X,Y}$  is the gradient of the Riemannian functional  $g \mapsto \int_M |W|^2 dV_g$ , acting on the space of all Riemannian metrics on  $M$ , cf. [10]. It thus follows that  $B$  is symmetric, traceless and conformally invariant (1,1)-tensor on  $M$ , cf. [10]; it vanishes if the metric  $g$  is (locally) conformal to Einstein by means of the second Bianchi identity (12), or if  $g$  is self-dual or anti-self-dual by means of the following expression for  $B$ , obtained by Gauduchon [24]:

$$\frac{1}{2} B_{X,Y} = \sum_{i=1}^4 [\nabla_{e_i}(\delta W^\pm)(X, e_i, Y) + W^\pm(X, e_i, h(e_i), Y)]. \quad (16)$$

In fact, identifying freely the  $(2, 0)$ -tensors with the corresponding 1-forms with values in  $TM$  via the metric  $g$  and using the second Bianchi identity (12), we get from (15)

$$(*B)_{X,Y,Z} = d^\nabla(*C)_{X,Y,Z} + \sigma_{X,Y,Z}[(\circ W)_{X,Y,h(Z)}], \quad (17)$$

where  $*$  :  $\Lambda^k M \otimes TM \mapsto \Lambda^{4-k} M \otimes TM$  and  $d^\nabla : \Lambda^k M \otimes TM \mapsto \Lambda^{k+1} M \otimes TM$  are the Hodge operator and the Riemannian differential, respectively, acting on the bundle of the  $k$ -forms with values in  $TM$ ;  $C = -d^\nabla h$  is the Cotton-York tensor, viewed as a section of  $\Lambda^2 M \otimes TM$ ;  $\circ W = W^+ - W^-$  is considered as a section of  $(\Lambda^2 M \otimes \Lambda^1 M) \otimes TM$  and  $\sigma_{X,Y,Z}$  denotes the cyclic sum on  $X, Y, Z$ .

On the other hand, on any Riemannian manifold we have

$$d^\nabla C_{X,Y,Z} = -(d^\nabla)^2 h_{X,Y,Z} = -\sigma_{X,Y,Z}[R_{X,Y,h(Z)}],$$

hence, using (2) we infer

$$d^\nabla C_{X,Y,Z} + \sigma_{X,Y,Z}[W_{X,Y,h(Z)}] = 0,$$

which, together with (17), implies

$$\begin{aligned} (*B)_{X,Y,Z} &= 2(d^\nabla(C^+)_{X,Y,Z} + \sigma_{X,Y,Z}[W_{X,Y,h(Z)}^+]) \\ &= -2(d^\nabla(C^-)_{X,Y,Z} + \sigma_{X,Y,Z}[W_{X,Y,h(Z)}^-]). \end{aligned} \quad (18)$$

Relation (16) follows now by applying the Hodge operator  $*$  to both sides in (18).

The Bach tensor can be also expressed in terms of the Ricci tensor  $Ric$  as follows: Using the second Bianchi identity (12), relation (15) can be rewritten as

$$B = \delta^\nabla d^\nabla h - \overset{\circ}{W}(h), \quad (19)$$

where:

- $h$  is considered as a section of  $\Lambda^1 M \otimes TM$ ;
  - $\delta^\nabla : \Lambda^{k+1} M \otimes TM \mapsto \Lambda^k M \otimes TM$  is the formal adjoint of  $d^\nabla$ ;
  - $\overset{\circ}{W}(t) = -\sum_{i=1}^4 W(X, e_i, t(e_i), Y)$  is the action of  $W$  on  $\Lambda^1 M \otimes TM$ .
- On the other hand, the Weitzenböck formula for the 1-forms with values in  $TM$  reads as [15, Proposition 4.1]

$$(\delta^\nabla d^\nabla + d^\nabla \delta^\nabla)t = \nabla^* \nabla t + t \circ Ric - \overset{\circ}{R}(t),$$

where  $t$  is a section of  $\Lambda^1 M \otimes TM$  and  $\overset{\circ}{R}(t) = -\sum_{i=1}^4 R(X, e_i, t(e_i), Y)$ . Using the above formula, specified for  $h = \frac{s}{24}g + \frac{1}{2}Ric_0$ , together with the Ricci identity  $\delta^\nabla h = -\frac{ds}{6}$ , we get from (19):

$$B = \nabla^* \nabla h + \frac{1}{6} \nabla ds + h \circ Ric - \overset{\circ}{W}(h) - \overset{\circ}{R}(h). \quad (20)$$

If  $(M, g, J)$  is an almost Hermitian 4-manifold with  $J$ -invariant Ricci tensor, then the traceless Ricci tensor has eigenvalues  $(\lambda, \lambda, -\lambda, -\lambda)$  and hence

$$h \circ Ric = \left( \frac{s^2}{96} + \frac{|Ric_0|^2}{8} \right) g + \frac{s}{6} Ric_0,$$

and by (2)

$$\overset{\circ}{R}(h) = \left( \frac{s^2}{96} + \frac{|Ric_0|^2}{8} \right) g + \overset{\circ}{W}(h).$$

Thus, (20) reduces to (see also [18, (24) and Lemma 4,(i)]):

$$B = \frac{1}{2} \nabla^* \nabla Ric_0 + \frac{\Delta s}{24} g + \frac{1}{6} \nabla ds + \frac{s}{6} Ric_0 - \overset{\circ}{W}(Ric_0). \quad (21)$$

If moreover  $(M, g, J)$  is an *almost Kähler* 4-manifold, we use (21) to obtain the following formula for  $B$ , which has a close resemblance with the expression of the Bach tensor on a 4-dimensional Kähler manifold obtained in [18].

**Lemma 3.** *Let  $(M, g, J)$  be an almost Kähler 4-manifold with  $J$ -invariant Ricci tensor. Then the Bach tensor  $B$  of  $g$  is given by*

$$B = -\frac{1}{3}(\nabla ds)^{inv} + \frac{1}{6}(\nabla ds)^{anti} - \frac{\Delta s}{12} g - \frac{s}{6} Ric_0 + S,$$

where:

- $(\nabla ds)^{inv}$  and  $(\nabla ds)^{anti}$  denote the  $J$ -invariant and  $J$ -anti-invariant parts of the symmetric  $(2, 0)$ -tensor field  $\nabla ds$ , respectively;
- $S$  is the symmetric,  $J$ -anti-invariant tensor, given by:

$$S = \sum_{i=1}^4 (\nabla_{e_i} \rho_0) \circ (\nabla_{e_i} \Omega),$$

with  $\rho_0(., .) = Ric_0(J., .)$  being the traceless Ricci form.

*Proof:* Since  $Ric_0$  is  $J$ -invariant, we have

$$\begin{aligned} \nabla^* \nabla Ric_0 &= -\nabla^* \nabla (\rho_0 \circ \Omega) \\ &= -(\nabla^* \nabla \rho_0) \circ \Omega - \rho_0 \circ (\nabla^* \nabla \Omega) + 2S. \end{aligned}$$

To compute  $\nabla^* \nabla \Omega$  and  $\nabla^* \nabla \rho_0$  we will use the Weitzenböck formula for 2-forms (also considered as sections of  $\Lambda^1 M \otimes TM$ ); for any 2-form  $\phi$  we have (cf. e.g. [12]):

$$\Delta \phi = \nabla^* \nabla \phi + \frac{s}{3} \phi - 2W(\phi).$$

Since the Kähler form  $\Omega$  is harmonic, we get

$$\rho_0 \circ (\nabla^* \nabla \Omega) = \frac{s}{3} Ric_0 + 2\rho_0 \circ W^+(\Omega). \quad (22)$$

Moreover, it is known that for an almost Kähler 4-manifold with  $J$ -invariant Ricci tensor, the Ricci form  $\rho = \frac{s}{4}\Omega + \rho_0$  is closed, see [20]. We thus compute

$$\delta \rho_0 = J(\delta Ric_0) = -\frac{Jds}{4},$$

$$d\rho_0 = -\frac{ds}{4} \wedge \Omega,$$

$$\Delta\rho_0 = -(\nabla ds)^{inv} \circ \Omega - \frac{\Delta s}{4} \Omega,$$

hence

$$(\nabla^* \nabla \rho_0) \circ \Omega = (\nabla ds)^{inv} + \frac{\Delta s}{4} g + \frac{s}{3} Ric_0 + 2W^-(\rho_0) \circ \Omega. \quad (23)$$

Since  $h = \frac{s}{24}g + \frac{1}{2}Ric_0$  is  $J$ -invariant, we have

$$\sum_{i=1}^4 W_3^+(X, e_i, h(e_i), Y) = 0.$$

Thus, using (6) and (7) we compute

$$\begin{aligned} \overset{\circ}{W}^+(h) &= \frac{1}{2} \overset{\circ}{W}^+(Ric_0) \\ &= \frac{\kappa}{12} Ric_0 - \frac{1}{8} [\mathcal{J}\Psi \circ Ric_0 - Ric_0 \circ \mathcal{J}\Psi]. \end{aligned}$$

As  $Ric_0$  is  $J$ -invariant, it anti-commutes with  $\mathcal{J}\Psi$  and the latter equality reduces to

$$\overset{\circ}{W}^+(Ric_0) = \frac{\kappa}{6} Ric_0 - \frac{1}{2} (\mathcal{J}\Psi) \circ Ric_0.$$

Using now (6), (7) and (24), we infer

$$2\rho_0 \circ W^+(\Omega) = \rho_0 \circ \left( \frac{\kappa}{3} \Omega - \Psi \right) = -2 \overset{\circ}{W}^+(Ric_0).$$

Similarly, considering instead of  $J$  the *negative* almost Hermitian structure  $\bar{J}$  that makes the Ricci tensor  $\bar{J}$ -invariant (at the points where  $Ric_0 \neq 0$ ), we get

$$2W^-(\rho_0) \circ \Omega = -2 \overset{\circ}{W}^-(Ric_0).$$

Hence, substituting (22) and (23) into (22), we eventually obtain

$$\frac{1}{2} \nabla^* \nabla Ric_0 = S - \frac{1}{2} (\nabla ds)^{inv} - \frac{\Delta s}{8} g - \frac{s}{3} Ric_0 + \overset{\circ}{W}^-(Ric_0),$$

and the claim follows by (21).  $\square$

**Corollary 2.** *Let  $(M, g, J, \Omega)$  be an almost Kähler 4-manifold with  $J$ -invariant Ricci tensor and constant scalar curvature. Suppose that the Bach tensor vanishes. Then either  $g$  is Einstein, or else the scalar curvature of  $g$  is zero.*

*Proof:* Considering the  $J$ -invariant part of  $B$  given by Lemma 3, we get  $sRic_0 = 0$  and the claim follows.  $\square$

**Remark 1.** Corollary 2 can be considered as a generalization of Corollary 1. Indeed, if  $(M, g, J)$  is an anti-self-dual almost-Kähler 4-manifold with  $J$ -invariant Ricci tensor, then the scalar curvature is constant by Lemma 1 and the Bach tensor vanishes by (16); moreover, it follows from (6), (8) and

(9) that the scalar curvature of an anti-self-dual almost Kähler metric  $g$  is zero if and only if  $(g, J)$  is Kähler.  $\diamond$

**Remark 2.** Using Lemma 3 in the compact case, a stronger result than Corollary 2 could be obtained. In fact *any compact almost Kähler 4-manifold with  $J$ -invariant Ricci tensor and non-positive scalar curvature, whose Bach tensor vanishes is either an Einstein manifold, or else it is a Kähler, scalar-flat surface.* Indeed, taking the inner product of the Bach tensor, as given in Lemma 3, with  $Ric_0$  and integrating over the manifold, we get:

$$0 = \int_M \left( -\frac{1}{3} \langle \nabla ds, Ric_0 \rangle - \frac{s}{6} |Ric_0|^2 \right) dV_g.$$

Integrating the first term by parts and taking into consideration that  $\delta Ric_0 = \frac{1}{4} ds$ , the above relation changes to:

$$0 = \int_M \left( \frac{|ds|^2}{12} - \frac{s}{6} |Ric_0|^2 \right) dV_g.$$

If  $g$  is *not* Einstein the above formula shows that  $s$  must identically vanish and then the conclusion follows from [19, Theorem 1].  $\diamond$

Note that both Corollaries 2 and Remark 2 apply for almost Kähler, self-dual manifolds with  $J$ -invariant Ricci tensor, as well.

**Remark 3.** It is well known that compact quotients of the complex hyperbolic space  $\mathbf{CH}_2$  admit Kähler-Einstein, self-dual metrics of negative scalar curvature. In the conformal class of such a metric strictly almost Kähler metrics could be found, as it follows from [5]. Thus, there are examples of self-dual, conformally Einstein, strictly almost Kähler metrics on compact 4-manifolds. On the other hand, a simple Bochner type argument shows that there are no strictly almost Kähler metrics in the conformal class of an anti-self-dual (equivalently, scalar-flat), Kähler metric (see *e.g.* [5]). However, it was pointed out by J. Armstrong [9] that strictly almost Kähler anti-self-dual metrics do exist on generic ruled surfaces. This follows from the analysis of the moduli spaces of anti-self-dual metrics and of scalar-flat, Kähler metrics done by King, Kotschick [30] (see also [28]) and LeBrun, Singer [36], respectively.  $\diamond$

### 3. THE SECOND CURVATURE CONDITION OF GRAY IN DIMENSION 4

In [25], A. Gray considered almost Hermitian manifolds whose curvature tensor has a certain degree of resemblance to that of a Kähler manifold. The following identities arise naturally:

$$G_1 \quad R_{XYZW} = R_{XYJZJW} ;$$

$$G_2 \quad R_{XYZW} - R_{JXJYZW} = R_{JXYJZW} + R_{JXYJZW} ;$$

$$G_3 \quad R_{XYZW} = R_{JXJYJZJW} .$$

We will call the identity  $G_i$  as the  $i$ -th condition on the curvature of Gray. It is a simple application of the first Bianchi identity to see that  $G_1 \Rightarrow G_2 \Rightarrow G_3$ . Also elementary is the fact that a Kähler structure satisfies relation  $G_1$

(hence, all of the relations  $G_i$ ). Following [25], if  $\mathcal{AK}$  is the class of almost Kähler manifolds, let  $\mathcal{AK}_i$  be the subclass of manifolds whose curvature satisfies identity  $G_i$ . We have the obvious inclusions

$$\mathcal{AK} \supseteq \mathcal{AK}_3 \supseteq \mathcal{AK}_2 \supseteq \mathcal{AK}_1 \supseteq \mathcal{K},$$

where  $\mathcal{K}$  denotes the class of Kähler manifolds. In [25] it was proven that the equality  $\mathcal{AK}_1 = \mathcal{K}$  holds locally (see also [26]), and it was also shown that the inclusion  $\mathcal{AK} \supset \mathcal{AK}_3$  is strict. The examples of Davidov and Muškarov [17], multiplied by compact Kähler manifolds, show that even in the compact case, the inclusion  $\mathcal{AK}_2 \supset \mathcal{K}$  is strict in dimension  $2n \geq 6$ .

In the compact, 4-dimensional case, it becomes apparent that the topology of the underlying manifold has consequences on the relationships between these classes. It was shown in [20, Theorem 3] that for a compact 4-manifold with second Betti number equal to 1, the equality  $\mathcal{AK}_3 = \mathcal{K}$  holds. We will now deal with almost Kähler 4-dimensional manifolds satisfying the second curvature condition of Gray.

First, let us observe that the condition  $G_2$  can be expressed in terms of the  $U(2)$ -decomposition (5) of the curvature as follows:

**Lemma 4.** *An almost Hermitian 4-manifold  $(M, g, J, \Omega)$  satisfies the second curvature condition of Gray if and only if the Ricci tensor is  $J$ -invariant,  $W_2^+ = 0$  and  $W_3^+ = 0$ .*

*Proof:* Easy consequence of (5), see [50].  $\square$

**Proposition 1.** *For an almost Hermitian 4-manifold satisfying the condition  $G_2$ , we have*

$$d(s - s^*) - \kappa\theta = 4Ric_0(\theta).$$

*Proof:* Using Lemma 4, the positive Weyl tensor of an almost Hermitian 4-manifold of the class  $\mathcal{AH}_2$  is given by (6). Thus, computing directly,

$$\alpha = -\frac{1}{2}J \langle \delta W^+, \Omega \rangle = -\frac{1}{8}\kappa\theta - \frac{1}{12}d\kappa.$$

From this, using Lemma 1 and (9), we obtain the identity claimed.  $\square$

**Remark 4.** It follows from the Riemannian version of the Robinson-Shild theorem in General Relativity that for the class of Hermitian 4-manifolds, the condition that  $Ric$  is  $J$ -invariant is in fact equivalent to the condition  $G_2$ , [4, Theorem 2]. Thus, Proposition 1 holds for any Hermitian surface of  $J$ -invariant Ricci tensor.  $\diamond$

Since on an almost Kähler 4-manifold, the 1-form  $\theta$  vanishes, we have the following immediate consequence of Proposition 1.

**Corollary 3.** *Let  $(M, g, J, \Omega)$  be a 4-dimensional almost Kähler manifold that satisfies the condition  $G_2$ . Then  $|\nabla J|^2 = 2(s^* - s)$  is a constant.*

It follows from Corollary 3 and (9), that if  $(M, g, J, \Omega)$  is almost Kähler, non-Kähler 4-manifold in the class  $\mathcal{AK}_2$ , then the Kähler nullity

$$\mathcal{D} = \{X \in TM \mid \nabla_X J = 0\}$$

of  $(g, J)$  is a well-defined 2-dimensional distribution over  $M$ . If we denote  $\bar{M}$  the manifold  $M$  with the reversed orientation, then we may consider the  $g$ -orthogonal almost complex structure  $\bar{J}$  on  $\bar{M}$ , defined in the following manner:  $\bar{J}$  coincides with  $J$  on  $\mathcal{D}$  and  $\bar{J}$  is equal to  $-J$  on  $\mathcal{D}^\perp$ . Denote by  $\bar{\Omega}$  the fundamental form of  $(g, \bar{J})$  and by  $W_i^-, i = 1, 2, 3$  the  $U(2)$  components of the negative Weyl tensor  $W^-$  determined by  $(g, \bar{J})$ . Then we have the following:

**Proposition 2.** *Let  $(M, g, J, \Omega)$  be a 4-dimensional almost Kähler, non-Kähler manifold, satisfying the condition  $G_2$ . Then*

(i) *The traceless Ricci tensor  $Ric_0$  of  $g$  is given by*

$$Ric_0 = \frac{\kappa}{4}[-g^{\mathcal{D}} + g^{\mathcal{D}^\perp}],$$

*where  $g^{\mathcal{D}}$  (resp.  $g^{\mathcal{D}^\perp}$ ) denotes the restriction of  $g$  on  $\mathcal{D}$  (resp. on  $\mathcal{D}^\perp$ ).*

(ii) *The triple  $(g, \bar{J}, \bar{\Omega})$  is an almost Kähler structure on  $\bar{M}$  with  $\bar{J}$ -invariant Ricci tensor and  $W_2^- = 0$ . Moreover,  $\mathcal{D}^\perp$  belongs to the Kähler nullity of  $(g, \bar{J}, \bar{\Omega})$ .*

*Proof:* (i) With the notations of Section 2.3, let  $\phi$  be a non-vanishing (local) section of  $\Lambda^{anti} M$ , which satisfies  $|\phi|^2 = 2$  at any point. As we have already mentioned, for an almost Kähler 4-manifold, the covariant derivative  $\nabla\Omega$  of the Kähler form  $\Omega$  can be written as

$$\nabla\Omega = a \otimes \phi - Ja \otimes \mathcal{J}\phi, \quad (24)$$

where the 1-form  $a$  is of constant length  $\frac{s^*-s}{4}$  (see (9) and Corollary 3) and  $\{a, Ja\}$  span  $\mathcal{D}^\perp$ . It follows from Lemma 2 that  $\beta = -\frac{1}{2}Ric_0(a)$ . Moreover, if the manifold also satisfies the condition  $G_2$ , i.e.,  $W_2^+ = 0, W_3^+ = 0$  (see Lemma 4), then by using (6) and (24) we directly compute:

$$\beta = \frac{1}{2}(-J \langle \delta W^+, \phi \rangle + \frac{1}{2} \phi \langle \delta W^+, \Omega \rangle) = -\frac{\kappa}{8}a.$$

Comparing the two expressions for  $\beta$  we get the first part of Proposition 2.

(ii) From the Ricci identity we get

$$(\nabla_{X,Y}^2 - \nabla_{Y,X}^2)(\Omega)(Z, T) = -R_{XYJZT} - R_{XYZJT}. \quad (25)$$

On the other hand, it follows from (24) that

$$\nabla^2|_{\Lambda^2 M} \Omega = (da - Ja \wedge \xi) \otimes \phi - (d(Ja) + a \wedge \xi) \otimes \mathcal{J}\phi, \quad (26)$$

where the 1-form  $\xi$  is defined from the equality  $\nabla\phi = -a \otimes \Omega + \xi \otimes \mathcal{J}\phi$ . We thus obtain from (25) and (26)

$$da - Ja \wedge \xi = -R(\mathcal{J}\phi); \quad d(Ja) + a \wedge \xi = -R(\phi).$$

As our manifold satisfies the condition  $G_2$ , we easily deduce from Lemma 4, (5) and (8) that  $R(\phi) = \frac{(s-s^*)}{8}\phi$  and  $R(\mathcal{J}\phi) = \frac{(s-s^*)}{8}\mathcal{J}\phi$ , hence the above

equalities reduce to

$$da = Ja \wedge \xi + \frac{(s^* - s)}{8} \mathcal{J}\phi; \quad d(Ja) = -a \wedge \xi + \frac{(s^* - s)}{8} \phi. \quad (27)$$

We thus get from (27) that  $d(a \wedge Ja) = 0$  and using the fact that  $|a|^2 = \frac{(s^* - s)}{4} = \text{const}$ , we conclude that the Kähler form  $\bar{\Omega} = \Omega - \frac{2}{|a|^2} a \wedge Ja$  is closed, i.e.,  $(g, \bar{J})$  is almost Kähler.

The Ricci tensor is  $\bar{J}$ -invariant because of Proposition 2, (i).

We shall further use the following implications, which are immediate consequences of (25):

- (a)  $\nabla^2|_{\Lambda-M}\Omega = 0$  if and only if the Ricci tensor is  $J$ -invariant;
- (b)  $\nabla^2|_{\Omega}\Omega = 0$  if and only if  $W_2^+ = 0$ ;
- (c)  $(\nabla_{Z_1, Z_2}^2 - \nabla_{Z_2, Z_1}^2)(\Omega)(Z_3, Z_4) = 0, \forall Z_1, Z_2, Z_3, Z_4 \in T^{1,0}M$  if and only if  $W_3^+ = 0$ .

To see that  $W_2^- = 0$ , we will prove first that  $(\nabla_X \bar{J}) = 0$  for any vector  $X$  from  $\mathcal{D}^\perp$ . Indeed, put  $Z_1 = A - iJA, Z_{\bar{1}} = A + iJA; Z_2 = B - iJB, Z_{\bar{2}} = B + iJB$ , where  $\{A, JA\}$  is an orthonormal frame of  $\mathcal{D}^\perp$  so as  $A$  and  $JA$  to be the dual orthonormal frame of  $\{a, Ja\}$ , and  $\{B, JB\}$  is an orthonormal frame of  $\mathcal{D}$ . Since  $(g, \bar{J})$  is almost Kähler, it is sufficient to prove that  $(\nabla_{Z_1} \bar{\Omega})(Z_1, Z_{\bar{2}}) = 0$ . Because  $(\nabla_{Z_1} \Omega)(Z_1, Z_{\bar{2}}) = 0$  this is equivalent to  $\nabla_{Z_1}(a \wedge Ja)(Z_1, Z_{\bar{2}}) = 0$ ; the latter equality follows from (27). Now, using that the Ricci tensor is also  $\bar{J}$ -invariant and the fact that  $\{A, JA\}$  is involutive, we easily conclude

$$\nabla^2|_{\bar{\Omega}}\bar{\Omega} = \nabla^2|_{\Omega}\bar{\Omega} + 2(\nabla_{A, JA}^2 - \nabla_{JA, A}^2)\bar{\Omega} = 0.$$

Thus we obtain  $W_2^- = 0$  by the corresponding version of the equivalence (b) written with respect to the negative almost Kähler structure  $\bar{\Omega}$ .  $\square$

We are ready now to prove Theorem 1.

**Proof of Theorem 1:** Let us assume that  $(M, g, J, \Omega)$  is a compact, strictly almost Kähler 4-dimensional manifold, satisfying the second curvature condition of Gray. Then by Corollary 3 we know that  $s^* - s$  is a non-zero constant function. Armstrong [7] showed that a compact 4-dimensional almost Kähler manifold with  $s^* - s$  nowhere vanishing, satisfies the topological condition:

$$2c_1^2(M) + c_2(M) = 0. \quad (28)$$

Equivalently we have,

$$5\chi(M) + 6\sigma(M) = 0, \quad (29)$$

since  $c_1^2(M) = 2\chi(M) + 3\sigma(M)$  and  $c_2(M) = \chi(M)$ , where  $\chi(M)$  and  $\sigma(M)$  are the Euler number and the signature of  $M$ , respectively.

We first prove that the Euler number and the signature are both zero under



our assumptions. By Proposition 2, the manifold with the reversed orientation  $\bar{M}$  also admits a symplectic form  $\bar{\Omega}$ . We use some consequences of the Seiberg-Witten theory. There are several cases, depending on the values of the Betti numbers  $b^+(M)$ ,  $b^-(M)$ , where  $b^+(M)$  ( $b^-(M)$ ) is equal to the dimension of the space of harmonic (anti-)self-dual 2-forms. It is a classical result that  $b^+(M)$ ,  $b^-(M)$  do not depend on the metric and they are in fact topological invariants of the compact 4-manifold.

(a)  $b^+(M) > 1$ ,  $b^-(M) > 1$

In this case we first remark that  $M$  and  $\bar{M}$  are minimal symplectic manifolds. Indeed, as shown in [33], for example, it follows that there are no embedded spheres of self-intersection  $\pm 1$  because the Seiberg-Witten invariants are non-vanishing on both  $M$  and  $\bar{M}$ , [47]. By a result of Taubes [48], a minimal compact symplectic 4-manifold  $(M, \Omega)$  with  $b^+(M) > 1$  satisfies

$$c_1^2(M) \geq 0.$$

Thus, we have  $c_1^2(M) \geq 0$  and  $\bar{c}_1^2(\bar{M}) \geq 0$ , where  $\bar{c}_1$  denotes the first Chern class of  $(\bar{M}, \bar{J})$ . In terms of the Euler class and the signature of  $M$ :

$$2\chi(M) + 3\sigma(M) \geq 0, \quad 2\chi(M) - 3\sigma(M) \geq 0. \quad (30)$$

Relations (29) and (30) imply  $\chi(M) = \sigma(M) = 0$ , which is equivalent to  $c_1^2(M) = c_2(M) = 0$ .

(b)  $b^+(M) = 1$ ,  $b^-(M) = 1$

Then, trivially,  $\sigma(M) = b^+(M) - b^-(M) = 0$  and we conclude from (29) that  $c_1^2(M) = c_2(M) = 0$  again.

(c)  $b^+(M) = 1$ ,  $b^-(M) > 1$

We will show that this case cannot occur. If  $c_1^2(M) = 2\chi(M) + 3\sigma(M) \geq 0$ , this leads to a contradiction, taking into account relation (29) and the fact that the signature has to be negative. If  $c_1^2(M) < 0$ , first remark that  $(M, \Omega)$  must be a minimal symplectic 4-manifold, by the argument invoked above. A. Liu [41] shows that a minimal symplectic manifold with  $b^+(M) = 1$  and  $c_1^2(M) < 0$ , is an irrational ruled surface. This is a contradiction since the signature of any ruled surface is 0, but with our assumptions  $\sigma(M) = 1 - b^-(M) < 0$ .

Now we prove that  $(\bar{M}, g, \bar{J}, \bar{\Omega})$  is, in fact, a Kähler structure. Using the local considerations proved in Proposition 2 (ii),  $(\bar{M}, g, \bar{J}, \bar{\Omega})$  is an almost Kähler structure and  $(\nabla_X \bar{J}) = 0$  for any vector  $X$  from  $\mathcal{D}^\perp$ , where  $\mathcal{D}$  is the Kähler nullity of  $(g, J)$ . Let  $B, JB$  be an orthonormal basis for the Kähler nullity  $\mathcal{D}$  of  $J$  and  $A, JA$  an orthonormal basis for  $\mathcal{D}^\perp$ . Consider on  $(M, g, J)$  the *first canonical connection*  $\nabla^0$ , defined by Lichnerowicz in [40] to be

$$\nabla_X^0 Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)(Y).$$

This is a Hermitian connection, so its Ricci form  $\gamma^0$  is, up to a constant, a representative for the first Chern class of  $(M, J)$ . A short computation of

$\gamma^0$  gives for any almost Hermitian manifold (see [27]):

$$\gamma^0(X, Y) = 4Ric^*(JX, Y) + \langle J\nabla_X J, \nabla_Y J \rangle.$$

The condition  $W_2^+ = 0$  is equivalent to the tensor  $Ric^*$  being symmetric (see (7)). Since we also assume that the Ricci tensor is  $J$ -invariant, it then follows that  $Ric^*$  and  $Ric$  have common traceless part (see (5)). Using now Proposition 2.(i), the above expression for  $\gamma^0$  simplifies in our case to:

$$\gamma^0 = (s + \kappa)A \wedge JA + (s^* - \kappa)B \wedge JB.$$

Therefore,

$$c_1^2(M) = \frac{1}{32\pi^2} \int_M (s + \kappa)(s^* - \kappa) dV_g. \quad (31)$$

Concerning  $(\bar{M}, g, \bar{J})$ , we obtain similarly

$$\bar{\gamma}^0 = -(\bar{s}^* + \kappa)A \wedge JA + (s - \kappa)B \wedge JB,$$

hence

$$\bar{c}_1^2(\bar{M}) = \frac{1}{32\pi^2} \int_M (\bar{s}^* + \kappa)(\kappa - s) dV_g, \quad (32)$$

where  $\bar{s}^*$  is the star-scalar curvature of  $(g, \bar{J})$ . Taking into account that  $c_1^2(M) = \bar{c}_1^2(\bar{M}) = 0$  and

$$\kappa - s = -3(s^* - \kappa) = \frac{3}{4}|\nabla J|^2 = \text{const} > 0,$$

we obtain from (31) and (32)  $\int_M (\bar{\kappa} - s) dV_g = 0$ . Thus,  $(g, \bar{J})$  is Kähler according to (9).

Moreover, from (31) we get  $\int_M (s + \kappa) dV_g = 0$ , which implies  $\int_M s dV_g < 0$  (see (9)). It is well known that the anti-canonical bundle of a compact Kähler surface of negative total scalar curvature has no holomorphic sections; then  $(\bar{M}, \bar{J})$  is either a ruled surface with base of genus at least 2, or else the Kodaira dimension is at least 1 (see for example [11]). If  $(\bar{M}, \bar{J})$  is a ruled surface, then it is minimal, since  $\sigma(\bar{M}) = 0$ , and hence  $\chi(\bar{M}) < 0$ , a contradiction. Moreover, the Kodaira dimension of  $(\bar{M}, \bar{J})$  could not be 2 since any surface of general type has positive Euler number. We conclude that  $(\bar{M}, \bar{J})$  belongs to class VI.  $\square$

#### 4. POINTWISE CONSTANT TOTALLY-REAL SECTIONAL CURVATURE

Let  $(M, g, J, \Omega)$  be an almost Hermitian manifold. As we have already mentioned in the introduction, a two-plane  $\sigma = X \wedge Y$  in the tangent bundle  $TM$  is said to be *totally-real* if  $J\sigma = JX \wedge JY$  is orthogonal to  $\sigma$ . The almost Hermitian manifold has *pointwise constant totally-real sectional curvature* if at any point the sectional curvature of the metric  $g$  is the same on all totally-real planes at that point.

The condition that  $\sigma = X \wedge Y$  is a totally-real plane is clearly equivalent to  $g(JX, Y) = 0$ , i.e., to  $\sigma$  being a Lagrangian plane with respect to the

fundamental 2-form  $\Omega$ . This allows us to obtain the following simple characterization of the 4-dimensional almost Hermitian manifolds of pointwise constant totally-real sectional curvature in terms of the  $U(2)$ -decomposition of the curvature tensor:

**Lemma 5.** ([23],[3]) *An almost Hermitian 4-manifold  $(M, g, J)$  is of pointwise constant totally-real sectional curvature if and only if the Ricci tensor is  $J$ -invariant,  $W_3^+ = 0$ ,  $W^- = 0$ . The totally-real sectional curvature  $\mu$  is given in this case by  $\mu = \frac{2s-\kappa}{24}$ .*

*Proof:* At any point  $x \in M$  we denote by  $\mathcal{J}_x^\perp$  the space of all positive Hermitian structures of  $(T_x M, g)$ , anti-commuting with  $J$ . The space  $\mathcal{J}_x^\perp$  can be identified with the elements of  $\Lambda_x^{anti} M$  of square norm 2, via the metric  $g$ . We start with the observation that a 2-plane  $\sigma = X \wedge Y$  in  $T_x M$  is totally-real with respect to  $J$  if and only if there exists  $I \in \mathcal{J}_x^\perp$  such that  $\sigma$  is a *holomorphic* plane with respect to  $I$ . Hence, the set of holomorphic planes for  $I$  when  $I$  varies in  $\mathcal{J}_x^\perp$  coincide with the set of totally-real planes for  $J$ . Thus,  $(M, g, J)$  is of pointwise constant totally-real sectional curvature if and only if for any  $I \in \mathcal{J}_x^\perp$ ,  $(M, g, I)$  is of pointwise constant holomorphic sectional curvature, say  $c$ , which does not depend on  $I$ . In particular we have that  $W^- = 0$  and  $Ric_0$  is  $I$ -anti-invariant for any  $I \in \mathcal{J}_x^\perp$ , cf. [32, 2]. The second condition is equivalent to the Ricci tensor (equivalently,  $Ric_0$ ) being  $J$ -invariant. Finally computing the holomorphic sectional curvatures for  $I$  and  $K = J \circ I$ , we obtain that they are equal if and only if  $W_3^+ = 0$ , i.e., if and only if  $R$  has the form:

$$R = \frac{s}{12} id + Ric_0^{inv} + \frac{\kappa}{8} \Omega \otimes \Omega - \frac{\kappa}{12} id^+.$$

Then for any totally-real 2-plane  $\sigma$  we have

$$\langle R(\sigma), \sigma \rangle = \frac{s}{12} |\sigma|^2 - \frac{\kappa}{12} |\sigma_+|^2 = \frac{(2s - \kappa)}{24} |\sigma|^2,$$

which shows that the totally-real sectional curvature is given by the formula claimed.  $\square$

**Proof of Theorem 2:** By Lemma 5 the Ricci tensor is  $J$ -invariant, so in the case on non-negative scalar curvature we apply [19, Theorem 1]. Suppose now that the scalar curvature is negative. Since  $g$  is self-dual (see Lemma 5), the Bach tensor vanishes (see (16)) and it follows from Corollary 2 that  $g$  is Einstein. Thus,  $(M, g, J, \Omega)$  is a But compact almost Kähler Einstein manifolds for which  $W_3^+ = 0$  (see Lemma 5). It must be Kähler according to [9, Theorem 4.3.4].  $\square$

**Remark 5.** Theorem 2 can be slightly generalized by assuming that the scalar curvature is everywhere non-negative or non-positive (but not necessarily constant). Indeed, in the non-negative case, the integrability of the almost Kähler structure follows by [19]. In the non-positive case by Remark 2 we have that either the scalar curvature identically vanishes, or else the metric is Einstein. If the scalar curvature identically vanishes, we apply

again [19], while in the latter case the integrability of the almost Kähler structure follows from [9]. We also observe that every compact almost Kähler 4-manifold of pointwise constant non-negative totally-real sectional curvature is Kähler. Indeed, let  $\mu$  be the totally-real sectional curvature. By (9) and (8), we have at any point,  $s \leq \kappa$ . In particular, we obtain from Lemma 5 the following pointwise inequality for the totally-real sectional curvature:  $\mu \leq \frac{s}{24}$ . Hence,  $\mu \geq 0$  implies  $s \geq 0$  at any point and the result follows from the observations above.  $\diamond$

As pointed out by Derdzinski [18], any Kähler metric in dimension 4 has a degenerate spectrum for the positive Weyl tensor, that is, the endomorphism  $W^+$  of  $\Lambda^+M$  has, at each point, at most two distinct eigen-values. Derdzinski also remarked that all known examples of Einstein metrics on compact, orientable 4-manifolds have degenerate spectrum for  $W^+$  (with one of the orientations of the manifold, at least). To our knowledge, this remark is still valid at this date. It is not hard to see that any compatible almost Kähler structure with an Einstein metric with degenerate spectrum for  $W^+$ , on a compact, *oriented*, 4-manifold, has to be, in fact, Kähler. (In other words, the Goldberg conjecture is true for such metrics.) Indeed, this follows essentially from the classification result of Derdzinski [18, Theorem 2], with touches of some more recent works in each of the three cases that occur:

- (i) [8, Theorem 2.4] or [44, Theorem 1] for the case of Einstein anti-self-dual metric;
- (ii) [5, Theorem 1] for the case when the metric admits a Kähler structure;
- (iii) [46] in the last case, when the scalar curvature is positive.

Regarding the degeneracy of the spectrum of the positive Weyl tensor for an almost Kähler 4-manifold of pointwise constant totally-real sectional curvature, we have the following

**Theorem 3.** *Let  $(M, g, J, \Omega)$  be a compact almost Kähler 4-manifold of pointwise constant totally-real sectional curvature. Then  $J$  is integrable if and only if the spectrum of the positive Weyl tensor is degenerate.*

*Proof:* It is known that for an almost Hermitian 4-manifold with  $W_3^+ = 0$ , the spectrum of  $W^+$  is degenerate if and only if  $W_2^+ = 0$  (see [45, 4]). Thus, according to Lemmas 4 and 5, for an almost Hermitian 4-manifold of pointwise constant totally-real sectional curvature, the spectrum of  $W^+$  is degenerate if and only if  $(g, J)$  satisfies the second curvature condition of Gray. If  $(M, g, J)$  is almost Kähler *non-Kähler* 4-manifold of pointwise constant totally-real sectional curvature, whose positive Weyl tensor is degenerate, it follows from Proposition 2 that there is an almost Kähler structure  $\bar{J}$ , compatible with  $g$  and the inverse orientation of  $M$ , such that the Ricci tensor of  $g$  is  $\bar{J}$ -invariant. As  $g$  is self-dual (Lemma 5), it follows from Corollary 1 that  $(g, \bar{J})$  is Kähler, hence  $g$  is scalar flat, contradiction to [19, Theorem 1]  $\square$

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